Design Proposal for Three Two-Dimensional Wind Tunnel Nozzles for Supersonic Probe Calibration

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Abstract

Although newer techniques such as laser Doppler and hot-wire anemometry have emerged, aerodynamic probes remain the best tools for the analysis of flows in turbomachines. Based on the nature of how these probes accomplish this analysis, they must be calibrated in well known flows prior to use. This calibration takes place in wind tunnels with uniform, parallel flows at precise Mach numbers.

The existing probe calibration wind tunnel at the division of Heat and Power Technology (HPT) at KTH has an operating range of Mach 0.1 to 0.95. This report seeks to provide guidance for the production of three new nozzles for this wind tunnel to raise the calibration range to Mach 1.2. The three nozzles will be designed to operate at Mach 1.0, 1.1, and 1.2 to allow for a range of probe calibration values.

This report will also include a cursory investigation into the loading effect of the supersonic flow on the probes to be calibrated.
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1 Introduction

Aerodynamic probes are widely used in turbomachines for flow measurement. The most basic probes have only one hole or “tap”. They are called Pitot probes and measure the total pressure of oncoming flow. These probes are often used as airspeed indicators in aviation applications. Probes used in turbomachinery are often more complex 3-hole and 5-hole varieties. The taps are located on different surfaces of the probe such that they encounter the flow at unique angles. An example probe diagram can be seen in Figure 12 in Appendix A. By measuring the variation between pressures on surfaces offset by known angles, a 2D or 3D flow description can be recorded. The way in which pressure measurements can be translated into a 3D flow description requires calibration coefficients - unique to each probe - to be known. These coefficients are obtained by testing probes in well regulated flow fields with uniform, parallel flow. These flow fields are produced by purpose built wind tunnel rigs. This report will provide guidance in the design of 3 nozzles for such a rig.

The existing probe calibration rig at HPT has a rectangular cross section and is fitted with a converging nozzle designed by Gomes.[1] This nozzle provides a jet of uniform, parallel flow at Mach numbers from 0.1 up to 0.95. In order to calibrate aerodynamic probes for testing at higher Mach numbers - up to 1.2 - three new nozzles must be designed to replace the existing one.

The velocity operating range of purely converging nozzles - for the establishment of subsonic or sonic flows - are governed by pressure ratios and therefore the same nozzle can be used for a large range of Mach numbers: up to \( \sim 1.0 \). Conversely, supersonic nozzle velocities are governed by area ratios between the throat and test section. This means that to produce two supersonic flows of different Mach number in the same test section, they must have a different throat area, or vice versa. Due to these phenomena, this report will provide guidance for the design of three separate nozzles. The first will be a converging nozzle, to produce Sonic flow at \( M=1.0 \) and the other two will be converging-diverging (Laval) nozzles to produce supersonic flow at \( M=1.1 \) and \( M=1.2 \), respectively.

In order to accomplish this task, a review of the scholarly literature on the subject of supersonic nozzle design was conducted. This investigation yielded myriad studies and information regarding the establishment of contours for the divergence of a Laval nozzle. The bases of these designs are methods for cancellation of expansion waves produced by the required wall curvature. Articles regarding the nozzle contractions were more difficult to come by, as the majority of converging nozzles are designed exclusively for the subsonic flow regime.

2 Conventions and Nomenclature

For convenience in reading the remainder of the report, some conventions will now be established to discuss the geometric considerations of the nozzles as well as the use of symbols.

First, the planned nozzles are all symmetric along a horizontal center line as depicted below in Figure 1. This centerline will act as the x-axis, with positive values to the right of the origin and negative values to the left. Several lengths along the x-axis shall be defined by the points 1-4 in Figure 1. A straight inlet duct spans a horizontal length of \( L_d \) between points 1 & 2. The contraction spans a horizontal length of \( L_c \) between points 2 & 3. Finally, the supersonic portion of the nozzle spans a horizontal length of \( L_s \) between points 3 & 4. Each of these lengths is dimensionless, and defined in relation to the throat half-height, \( h_t \). For example, \( L_d \cdot h_t \) would give the distance between points 1 & 2 in meters. The same is true of \( L_c \) and \( L_s \).
The y-axis is defined normal to the centerline at the point where it intersects the sonic line. Although the sonic line is never perfectly straight, that is the desired operating condition - and thus how it is represented in Figure 1. Due to the symmetry - both of the nozzle and corresponding Mach waves - it is simplest to define the contour by half-height, or distance from the centerline. Again for simplicity, the half-heights will be dimensionless and defined in relation to the throat half-height, $h_t$. Therefore, $y_t$ will be defined as unity. The other important $y$-values are defined as $y_i = h_i/h_t$ and $y_e = h_e/h_t$. The function describing the contour will be expressed as a function $y = f(x)$ such that:

$$y_i = f(-L_c), \quad y_t = f(0), \quad y_e = f(L_s).$$

Several of the sections of this report will discuss the method of characteristics. In this method, Mach lines emanating from the nozzle walls are called characteristics. For clarity, it should be noted that the terms Mach lines, characteristic lines, and expansion waves are interchangeable in this approach. Right-running characteristics ($\Psi_-$) emanate from the lower wall, travel in the $(+x,+y)$ direction, and cause the flow to turn clockwise. Left-running characteristics ($\Psi_+$) emanate from the upper wall, travel in the $(+x,-y)$ direction, and cause the flow to turn counter-clockwise. Along these lines, certain flow parameters are constant, by the compatibility equations:

$$\Psi_- = \nu - \theta$$
$$\Psi_+ = \nu + \theta$$

Wherein $\nu$ is the Prandtl-Meyer function given by:

$$\nu(M) = \sqrt{\frac{\gamma+1}{\gamma-1}} \tan^{-1} \sqrt{\frac{\gamma-1}{\gamma+1}(M^2-1)} - \tan^{-1} \sqrt{M^2-1}$$

(1)
### Table 1: Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>α</td>
<td>Mach angle, $\alpha = \sin^{-1}(1/M)$</td>
<td>$[^\circ]$</td>
</tr>
<tr>
<td>β</td>
<td>Probe deflection angle due to aerodynamic loading</td>
<td>[rad]</td>
</tr>
<tr>
<td>γ</td>
<td>Ratio of specific heats</td>
<td>[-]</td>
</tr>
<tr>
<td>θ</td>
<td>Angle between nozzle contour and centerline</td>
<td>$[^\circ]$</td>
</tr>
<tr>
<td>µ</td>
<td>Dynamic viscosity</td>
<td>[kg/m-s]</td>
</tr>
<tr>
<td>ν</td>
<td>Prandtl-Meyer angle</td>
<td>$[^\circ]$</td>
</tr>
<tr>
<td>ρ</td>
<td>Fluid density</td>
<td>[kg/m$^3$]</td>
</tr>
<tr>
<td>τ</td>
<td>Wind tunnel cross-section width, $\tau = 0.1$</td>
<td>[m]</td>
</tr>
<tr>
<td>Φ</td>
<td>Velocity Potential</td>
<td>[-]</td>
</tr>
<tr>
<td>Ψ⁻</td>
<td>Right-running characteristics</td>
<td>$[^\circ]$</td>
</tr>
<tr>
<td>Ψ⁺</td>
<td>Left-running characteristics</td>
<td>$[^\circ]$</td>
</tr>
<tr>
<td>c</td>
<td>Isentropic sound speed</td>
<td>[m/s]</td>
</tr>
<tr>
<td>C_D</td>
<td>Coefficient of Drag</td>
<td>[-]</td>
</tr>
<tr>
<td>E</td>
<td>Young’s modulus</td>
<td>[GPa]</td>
</tr>
<tr>
<td>h</td>
<td>Half-height of the nozzle</td>
<td>[m]</td>
</tr>
<tr>
<td>I</td>
<td>Moment of inertia</td>
<td>[kg-m$^2$]</td>
</tr>
<tr>
<td>m</td>
<td>Mass flowrate</td>
<td>[kg/s]</td>
</tr>
<tr>
<td>M</td>
<td>Mach number</td>
<td>[-]</td>
</tr>
<tr>
<td>O'</td>
<td>Origin of theoretical source flow in method of Foelsch</td>
<td>[-]</td>
</tr>
<tr>
<td>p</td>
<td>Static pressure</td>
<td>[kPa]</td>
</tr>
<tr>
<td>p₀</td>
<td>Stagnation or total pressure</td>
<td>[kPa]</td>
</tr>
<tr>
<td>ℛ</td>
<td>Ideal gas constant</td>
<td>[J/kg-K]</td>
</tr>
<tr>
<td>R</td>
<td>Radial distance from $O'$ normalized by $h_t$</td>
<td>[-]</td>
</tr>
<tr>
<td>Re</td>
<td>Reynolds’s number</td>
<td>[-]</td>
</tr>
<tr>
<td>T₀</td>
<td>Stagnation or total temperature</td>
<td>[K]</td>
</tr>
<tr>
<td>u</td>
<td>x-component of velocity</td>
<td>[m/s]</td>
</tr>
<tr>
<td>v</td>
<td>y-component of velocity</td>
<td>[m/s]</td>
</tr>
<tr>
<td>x</td>
<td>Distance along x-axis, normalized by throat half-height</td>
<td>[-]</td>
</tr>
<tr>
<td>y</td>
<td>Distance along y-axis, normalized by throat half-height</td>
<td>[-]</td>
</tr>
<tr>
<td>z</td>
<td>Distance along x-axis from $-L_c$ to 0, $z = x/L_c$</td>
<td>[-]</td>
</tr>
<tr>
<td>( )ᵢ</td>
<td>Condition at nozzle inlet</td>
<td>Various</td>
</tr>
<tr>
<td>( )ₜ</td>
<td>Condition at nozzle throat</td>
<td>Various</td>
</tr>
<tr>
<td>( )ₚ</td>
<td>Condition at point of inflection of nozzle</td>
<td>Various</td>
</tr>
<tr>
<td>( )ₑ</td>
<td>Condition at nozzle exit</td>
<td>Various</td>
</tr>
</tbody>
</table>

### 3 Background

An investigation of the scholarly literature regarding the topic of supersonic nozzles is always the initial step in the design process. There are two primary types of nozzle designs: two-dimensional or planar, and axisymmetric. Both types of nozzles can be represented in 2D diagrams. Two-dimensional or planar nozzles have rectangular cross sections and consist of their 2D representation extended out of the plane they are drawn on. Axisymmetric nozzles have circular cross sections and consist of their 2D representation rotated 180° about the centerline. The nozzles discussed here will be of the two-dimensional variety because the existing wind tunnel rig at HPT is of this geometry (with rectangular cross section). These nozzles are easier to manufacture and only present difficulties in flows with high stagnation pressure and temperature. In high $p₀$ and $T₀$ flows, axisymmetric nozzles relieve problems of dimensional instability.[3]

As mentioned before, the majority of the literature regarding supersonic nozzles is concerned with the contours of the diverging portion. The classic approach to the problem is the method developed by Prandtl and Busemann in 1929 involving the Method of Characteristics (MoC). The problem that arises in these diverging sections is characteristic of all supersonic flows; When the
nozzle cross-section initially expands after the throat, the flow is turned away from the centerline causing expansion waves to propagate down the remainder of the nozzle. Similarly, compression waves are created after the point of inflection (POI) of the nozzle contour, by turning the flow back towards the centerline. The Method of Characteristics seeks to establish the sets of waves in such a way that the effect of the compression waves serves to cancel out the effect of the reflected expansion waves. The MoC relies on the fact that flow properties are constant along Mach lines (expansion and compression waves) via the compatibility equations. The literature cited herein contains much information on the application of MoC to supersonic nozzles, and Shapiro\cite{4} devotes a whole chapter to the theory behind the application.

Figure 2 below shows the supersonic portion of a Laval nozzle with certain features valuable to the MoC. Characteristic Mach lines are depicted by both dashed and solid lines emanating from 7, where the sonic line intersects the upper nozzle contour. The characteristics below the centerline are not depicted, as there is symmetry about the centerline and depicting both is redundant. In this depiction, the right-running waves that would emanate from the mirror of 7 can simply be imagined as the reflections of the left-running waves off of a flat wall in place of the centerline.

Figure 2: Diagram of nozzle sections with characteristics\cite{4}

In this diagram, different sections of the nozzle are described by the solid characteristic lines. The expansion zone is bounded by the throat 6-7, the centerline 6-2, the nozzle wall 7-3, and the characteristic 3-2. This is the area in which both left and right-running characteristics exist, and the wall contour is concave up. The point of inflection of the wall contour is marked by 3, and the characteristic 3-2 is the expansion wave that accelerates the centerline of the flow to the desired test section Mach number, $M_e$. The test section is described by the final characteristic 2-1, the centerline, and the upper wall. Because the flow that passes through this final characteristic is uniform and parallel, the characteristic is a straight line and 1 lies at the point where $d\theta/dx = 0$, with $\theta$ representing the inclination of the nozzle wall. The upstream portion of this section is diamond shaped with a vertex half-angle, $\alpha_e$ which is equivalent to the Mach angle for the desired test section Mach number:

$$\alpha_e = \sin^{-1}(1/M_e)$$

The middle section is the straightening section, described by the wall contour 3-1 and the characteristics 3-2 and 2-1. In this region, only one family of waves exists. Because of the fact that 3 is a POI and $d\theta/dx$ reaches zero at 1, the wall contour in this section is concave down. Based on the geometric considerations, expansion waves are created by the wall curvature on the segment 7-3, the expansion section. These waves then reflect alternately off the centerline and wall until 3, when the wall curvature changes sign and no longer causes expansion waves. In order to create shock-free flow in the test section, the wall contour along 3-1 must be designed to cancel out the expansion waves that are incident on it. This is done by assigning a shape to the nozzle contour matching a hypothetical streamline that would result from having the flow
at 3 pass the expansion waves incident on the wall along 3-1. Shapiro suggests that given this logic, a straightening section can be properly designed given any expansion section. Therefore the expansion section of the nozzle should be designed first.

The original MoC approach of Prandtl and Busemann relied on tedious numerical and graphical methods to determine the contours. This method was adapted by Foelsch[5] in 1949 to provide an analytical solution. The basics of the MoC previously discussed all remain applicable in the method of Foelsch, except the requirement that the expansion section be designed first. He begins by assigning an imaginary radial source flow from a point $O'$ as depicted in Figure 8 in Section 5.1. The point $O'$ is positioned on the centerline so that it produces rays tangent to the nozzle contours at the points of inflection. The angle between these rays and the centerline is therefore $\theta_p = \theta_{max}$. This method therefore bases the nozzle contours on the location and wall inclination of the point of inflection, working both towards the throat and towards the test section from there. Thus Foelsch provides flow conditions at the POI first and using these boundary conditions, assigns contours for both the expansion and straightening sections separately.

The method of Foelsch, as presented in several sources[2,5,6,7] provides an analytical solution to the problem of nozzle contour design. For this reason, and the fact that the analytical solution is capable of providing quality results, this will be the method described in this report. More specifics and design considerations will be discussed in Section 5.

4 Sonic Nozzle

The first nozzle to be produced should be the converging nozzle, designed for sonic flow at its exit. The following section will discuss theoretical approach to designing such a nozzle. The approach is adapted from the research of Ho & Emanuel[8]. The original intent of their article was to determine contours for a nozzle contraction to be used in conjunction with a diverging portion, i.e. to produce uniform sonic throat flow. This design criterion allows the following contraction design to be implemented as both a sonic nozzle, and the first portion of a Laval nozzle. This section will therefore serve as design guidelines for the sonic nozzle and half of each supersonic nozzle; the next section will solely discuss the divergent portion of the supersonic nozzles.

4.1 Theory

This first subsection will deal with a theoretical approach to the design of the nozzle contraction. In the following subsection, preliminary calculations and other considerations will be made to assist in the adaptation of this theory to a physical nozzle.

The contraction operates in the subsonic range, and thus involves no pressure waves. Therefore, it is best to start by noting that the converging section is defined by several geometric parameters[8,9]:

- The initial half-height of the contour must match the half-height of the preceding duct
- The final half height must match the critical flow requirement of the throat ($M_t = 1.0$)
- The contour must decrease monotonically on the interval $x \in [-L_c, 0]$
- The contour must satisfy $\frac{d^n f}{dx^n} = 0$ at $x = -L_c$, for $n = 1,2,\ldots,m$
- The contour must satisfy $\frac{d^n f}{dx^n} = 0$ and $\frac{d^{n+1} f}{dx^{n+1}} \neq 0$ at $x = 0$, for $n = 1,2,\ldots,m$

Based on these requirements, any polynomial function to describe the nozzle contour would require infinite length, $L_c$ providing a trivial solution. Therefore Ho & Emanuel suggest that the contour follows an error function profile:

$$f(x) = a + b \cdot \text{erf}[\eta(z)], \quad x \in [-L_c, 0]$$  \hspace{1cm} (2)

where $\text{erf}[\eta(z)]$ is the error function of $\eta(z)$ and,

$$a = 1, \quad b = \frac{y_t - 1}{\text{erf}[\eta(-1)]}, \quad \eta(z) = \frac{z^{m+1}}{1 + (-1)^m(m + 1)z^{m+2}}$$  \hspace{1cm} (3)
therefore the contour is defined by:

\[
f(x) = 1 + (-1)^{m+1} \frac{y_i - 1}{\text{erf}[1/(m+2)]} \text{erf}\left(\frac{(x/L_c)^{m+1}}{1 + (-1)^m (m+1)(x/L_c)^{m+2}}\right)
\]

(4)

This contour is governed by a few independent variables: \(y_i, m, \text{ and } L_c\). The ratio of the initial half-height of the inlet duct to the throat half-height, \(y_i\) is a measurement of how much the nozzle contracts. The whole-number value \(m\) is expressed in the list of geometric parameters above, as a measure of the smoothness of the transition of the contour at the sonic exit/throat. Finally, \(L_c\) is the length that the contour spans, akin to a measurement of how gradual the contraction is. The impact of each of these parameters - \(L_c, m, \text{ and } y_i\) - are investigated by Ho & Emanuel using CFD techniques, namely the NPARC[6] code. Additionally, the length of the duct preceding the contraction, \(L_d\) is investigated for its effect on inlet Mach number uniformity.

First, simulations were run with the following baseline values:

\[
\gamma = 1.4, y_i = 4, m = 3, L_c = 4, L_d = 6
\]

Where all length parameters \((y_i, L_c, L_d)\) represent dimensionless length, normalized by \(h_t\). The results of the baseline test can be seen in Figure 3, below.

![Figure 3: Wall and centerline Mach number (a) and pressure ratio (b) [8]](image)

The results show two things. First, that the wall and centerline values reach Mach number and \(p/p_0\) of 1.0 and 0.5283, respectively at \(x = 0\) at the same time. This indicates preliminary justification that the nozzle accomplishes what it is designed to: produce uniform, sonic flow at the throat. Secondly - and also quite importantly - it shows that the gradients of both values are quite small near \(x = 0\) along both the centerline and wall. This second point is very beneficial for the design of a converging nozzle with sonic exit, as the emerging jet will have no tendency toward acceleration. Ho & Emanuel attribute the small gradient to the relatively high value of \(m = 3\). This high level of curvature control should therefore be maintained.

After the baseline evaluation, each value was varied independently while the other values were maintained at baseline. Figure 4 below shows the effect of \(L_d\) on the Mach number distribution at both ends of the contraction. Specifically, 4a depicts \(M_i(y)\) at \(x = -L_c\) and 4b depicts \(M_t(y)\) at \(x = 0\).
It can be seen from Fig. 4a, that a low value of $L_d$ causes a flow at the inlet that is fast along the centerline and slow at the walls. This lack of uniformity leads to large variability in Mach number at the throat, as shown in Fig. 4b. This lack of uniformity is highly undesirable. Ho & Emanuel found that for the baseline conditions mentioned before, a value of $L_d = 6$ was sufficient for nearly uniform flow at both ends of the contraction.

The other significant length parameter, $L_c$ was varied next, and the results of several simulations can be seen in Figure 5, below.

![Figure 5: Effect of contraction length ($L_c$) on centerline (a) and wall (b) Mach numbers][8]

**Figure 5**: Effect of contraction length ($L_c$) on centerline (a) and wall (b) Mach numbers

Fig. 5a shows that the length of the contraction has very little impact on the centerline exit Mach number. The primary trend that can be seen is that a lengthening of the contraction serves to lessen the Mach number gradient near the exit at $x = 0$. While this is a desirable effect, Fig. 5b reveals a much more significant impact of $L_c$. For low $L_c$ values, $M_e$ at the wall is extremely high, indicating a highly curved sonic line. This too, is undesirable for obvious reasons. The figure also indicates a large adverse pressure gradient, which leads to boundary layer separation. When tested with the other baseline values listed before, a $L_c = 5$ or greater produced a monotonic trend at both locations, and uniform flow at the exit plane.

The final parameters to be varied, were $m$ and $y_i$. These were done together and the results are presented in Figure 6 below.
These two parameters have less impact on the flow quality than those previously discussed. The figure indicates that increasing throat smoothness moves the steep portion of the gradient upstream, thereby producing more uniform flow at the throat. It can be seen that a value of $m = 3$ or $m = 4$ should be sufficient for uniform flow. The increase to $m = 4$ is mostly required for large values of $y_i \geq 6$. For reasonably small area ratios, $m = 3$ should be sufficient.

4.2 Application

In order to employ these contour design guidelines in the construction of a sonic wind tunnel, a few geometric considerations must be made. First, an adaptation to the nozzle should be made, by attaching the contraction at its throat to a straight section of sonic flow prior to the open walled test section. This will allow the probe to access a steady sonic flow instead of encountering the flow in either the open test section, or the contraction itself. An explanation of this geometry can be gleaned from Figure 6 below. It should be noted that unlike in the figure, the probe should ideally be placed exactly in the center of the flow. The second set of considerations are the pressure and mass flow requirements that constrain the nozzle size.

The existing tunnel into which the nozzle inserts shall be placed has a rectangular cross section
250mm tall ($y$-axis), and 100mm wide (out of the plane). The nozzle and test section will maintain the $\tau = 0.1m$ width, and only the height will be adjusted along the nozzle contours. The inlet half-height of the nozzle is also set at $h_i = 0.125m$. The throat height is therefore the adjustable quantity in determining the nozzle geometry. A definitive value for $h_t$ can be found in terms of the wind tunnel compressor operating conditions, namely $\dot{m}, p_0$, and $T_0$. Mass flowrate is defined as follows:

$$\dot{m} = \rho AV$$

Where

$$\rho = \frac{p_0}{RT_0} \left(1 + \frac{\gamma - 1}{2} M^2\right)^{-1/(\gamma - 1)}$$

$$A = 2h \cdot \tau$$

$$V = M \sqrt{\frac{\gamma}{RT_0}} \left(1 + \frac{\gamma - 1}{2} M^2\right)^{-1/2}$$

for compressible flow at Mach number $M$. At the sonic throat, where $M = M_t = 1$ and $h = h_t$, this combines to:

$$\dot{m} = 2h_t \tau p_0 \sqrt{\frac{\gamma}{RT_0}} \left(1 + \frac{\gamma - 1}{2}\right)^{-\frac{\gamma + 1}{2(\gamma - 1)}}$$

This can be rearranged to:

$$h_t = \frac{\dot{m}}{2h \tau p_0 \sqrt{\frac{\gamma}{RT_0}}} \left(1 + \frac{\gamma - 1}{2}\right)^{\frac{\gamma + 1}{2(\gamma - 1)}}$$

Eq. 10 provides the throat half height as a function of the variable rig operating conditions ($\dot{m}, p_0, T_0$), and the constants tunnel width, gas constant, and ratio of specific heats ($\tau, R, \gamma$). The operating condition of the rig should be set based on the required pressure ratio of the nozzle. The test section of the rig is left open to the atmosphere to allow the probe access to the flow. This sets the back pressure of the nozzle $p_b = p_a$, the ambient pressure in the test facility. Assuming isentropic flow, the exit pressure $p_e$ of the nozzle can be related to the stagnation pressure by:

$$\frac{p_0}{p_e} = \left(1 + \frac{\gamma - 1}{2} M_e^2\right)^{\gamma/(\gamma - 1)}$$

For ideal nozzle operation, the exit pressure of the nozzle exactly matches the back pressure. In this case the ideal condition is $p_e = p_b = p_a$ when $M_e = 1$. Therefore:

$$p_0 = p_a \left(1 + \frac{\gamma - 1}{2}\right)^{\gamma/(\gamma - 1)} \approx \frac{p_a}{0.58283}$$

Eq. 12 gives the desired operating condition based on the ambient pressure in the test room, which can be measured easily, and the constant $\gamma$. The ratio given is accurate when $\gamma = 1.4$.

In summary, the nozzle contours for the converging portion of all three nozzles should be governed by Eq. 4. The nozzle should be designed such that the variable parameters therein are:

$$L_c \geq 5, \; L_d \geq 6, \; y_i = \frac{125}{h_t}, \; \text{and} \; m = \left\{ \begin{array}{ll} 3, & 2 \leq y_i \leq 6 \\ 4, & 6 \leq y_i \end{array} \right.$$  

\section{5 Supersonic Nozzles}

In this section, design considerations for two Laval nozzles - capable of producing uniform, parallel flows of Mach 1.1, and 1.2 - will be presented. As mentioned in the previous section, the contraction forming the subsonic portion of the supersonic nozzles shall have the same contour
design as the sonic nozzle. The remaining portion of the nozzle has two design characteristics. First, the area ratio between the throat and exit planes of the nozzle are defined by certain isentropic flow considerations that will be explained more later. Secondly, the contours must be designed in such a way that the expansion waves created by turning the supersonic flow past the parallel throat, and the compression waves formed by turning the flow after the POI back towards the centerline have a zero sum effect on the flow in the exit plane.

5.1 Theory

The ideal design for the diverging portion of a Laval nozzle requires a straight sonic line, vertical across the throat. The contraction discussed in the previous section is designed specifically for this case. The method of characteristics, addressed briefly in Section 3 is primarily a way to determine the properties of a supersonic flow as it encounters solid surfaces. The specific application of the method with regards to nozzle design is to solve the inverse problem; for a desired flow, what physical boundaries are required? The method of Foelsch was briefly introduced in Section 3, but will be elaborated upon here. The figure below depicts the geometrical terminology of the method.

![Figure 8: The variables used in the method of Foelsch](image)

The boundary conditions of this method are well presented by Crown and Heybey: “(1) Along the Mach line emanating from the point of inflection \(p\), the velocity vectors are co-original, (2) The Mach number is constant along the arc of the circle that passes through the inflection point of the wall perpendicularly (and obviously its center is the origin \(O'\) of the velocity vectors), (3) in the region between this arc and the Mach line from the inflection point, the Mach number is a function solely of the radius \(R\) from the vector origin.”\(^2\) From these conditions, a hypothetical sonic radius \(R_0\) is established as expressed below. From this value all other radial distances \(R\) can be described as a function of Mach number by an area ratio:

\[
R_0 = \frac{y_t}{\theta_p} = \frac{1}{\theta_p} \tag{14}
\]

\[
R = R_0 \left( \frac{A}{A_0} \right) \tag{15}
\]

With

\[
\frac{A}{A_0} = \sqrt{\frac{1}{M_e^2} \left[ \left( \frac{2}{\gamma + 1} \right) \left( 1 + \frac{2}{\gamma - 1} M_e^2 \right) \right]^{\gamma + 1/(\gamma - 1)}}
\]

From the figure, the point of inflection has coordinates\((x_p, y_p)\) and the Mach line coincident with it has all along its length the coordinates\((x_2, y_2)\). The length \(l\) in the figure represents the
Mach line between the POI Mach line and the point on the nozzle contour where the expansion wave is canceled out. The values of $x_p$, $y_p$ and $l$ are given by:

$$x_p = R_p \cos \theta_p$$

$$y_p = \frac{R_p \sin \theta_p}{A_p}$$

$$l = M R (\nu - \nu_p) a$$

For parallel flow in the exit plane of the nozzle, $\theta_e = 0$. Therefore the compatibility equation reduces and $\nu_p$ can be expressed as follows:

$$(\Psi_+)_e = (\Psi_+)_p$$

$$\nu_e + \nu_p = \nu_p + \theta_p$$

$$\nu_p = \nu_e - \theta_p$$

The POI Mach line can be described by the geometry as:

$$x^2 - x_p = -R_p \cos \theta_p + R \cos(\nu_e - \nu)$$

$$y^2 = R \sin(\nu_e - \nu)$$

These POI mach line coordinates represent the starting points of the characteristic lines of length $l$. With the expression for $l$ given in Eq. 17 the location of the endpoints - the nozzle wall coordinates - can be determined:

$$x - x_p = x_2 - x_p + l \cos(\nu_e - \nu - \alpha_e)$$

$$y = y_2 + l \sin(\nu_e - \nu - \alpha_e)$$

The advantage of the method of Foelsch is clearly that it provides an analytical solution. Unfortunately this solution often requires a discontinuity in curvature at the POI. An example of the scale of the discontinuity can be seen in Figure 15 in Appendix A. This discontinuity can cause the nozzle to be fickle in that a small change in operating conditions can cause the design to be ineffectual. According to Goffert et al.\cite{7} small discontinuities do not necessarily cause significant error in Mach number distribution, and therefore the method should not be discounted. Some work has been done on developing nozzle contours of continuous curvature, specifically by Evvard & Marcus\cite{11}, and also J.C. Sivells. They provide auxiliary boundary conditions to ensure the continuity of nozzle wall curvature.

### 5.2 Application

The wall contour for the straightening section is determined by Eq. 19a and 19b as a function of several angles: $\nu$, $\theta_p$, $\nu_e$ and $\alpha_e$. The variable angle $\nu$ is a function of local Mach number and changes based on location. Basically this means that the contour equations are given indirectly as a function of $M$ and several constants. The latter two angles $\nu_e$ and $\alpha_e$ are conditions of the exit Mach number and can be determined from earlier equations, assuming $\gamma = 1.4$:

$$\nu_e = \begin{cases} 1.3362^\circ, & M_e = 1.1 \\ 3.5582^\circ, & M_e = 1.2 \end{cases}$$

$$\alpha_e = \sin^{-1} \left( \frac{1}{M_e} \right) = \begin{cases} 65.380^\circ, & M_e = 1.1 \\ 56.443^\circ, & M_e = 1.2 \end{cases}$$

Determining the maximum wall inclination $\theta_p$ is a slightly more difficult task. The A Priori assumption is often made that $\theta_p = \nu_e/2$. This assumption is tested by Goffert et al.\cite{7} with a nozzle designed by the same method of Foelsch for a test section Mach number $M_e = 1.3$. In the report, the ratio of $\nu_e/\theta_p$ is varied from 2 to 2.6. The results can be seen below in Figure 9 (the ratio $\nu_{TS}/\theta_{inf}$ in the legend is equivalent to $\nu_e/\theta_p$).
A ratio of $\theta_p = \nu_e/2.6$ provided the most uniform test section Mach number. For the nozzles of this report according to this ratio, and assuming $\gamma = 1.4$:

$$\theta_p = \frac{\nu_e}{2.6} = \begin{cases} 0.5139^\circ, & M_e = 1.1 \\ 1.3685^\circ, & M_e = 1.2 \end{cases}$$

(22)

The preceding analysis explains how the Method of Foelsch may be used to construct the straightening portion of the nozzle, but how should the expansion zone be established? There is no definitive reasoning behind determining a shape for this portion of the nozzle. There are several extant choices in the literature, some have proposed a simple arc$^{[5,9]}$, or a cubic polynomial$^{[2]}$. Goffert et al. tested these two methods against a third, involving a 4th order polynomial of the form:

$$y = Ax^4 + Bx^3 + Cx^2 + y_t$$

The results of the trials can be seen below in Figure 10.
The 4th order polynomial here shows the least variation in Mach number of the three design options. The coefficient $A$ was tuned to match the straightening curve at the POI and has a value of $A = -0.4472$. The coefficients $B$ and $C$ are then given by:

$$B = \frac{y_p - 1}{1.5x_p^3} + \frac{2}{3}Ax_p^2 - \frac{3\tan\theta_p}{x_p^2}$$

$$C = \frac{\tan\theta_p}{2x_p} - 2Ax_p^2 - 1.5Bx_p$$

The analysis presented in Section 4.2 remains a valid method for determining the throat area. Eq. 10 provides the throat half-height as a function of the flow parameters $\dot{m}, p_0$, and $T_0$:

$$h_t = \frac{\dot{m}}{2\pi p_0} \sqrt{\frac{RT_0}{\gamma}} \left(1 + \frac{\gamma - 1}{2}\right)^{\frac{\gamma+1}{\gamma-1}}$$

Two new calculations can be carried out using Eq. 11 and plugging in (a) $M_e = 1.1$ and (b) $M_e = 1.2$ to produce for the two Laval nozzles:

$$p_0 = p_a \left(1 + \frac{\gamma - 1}{2}(1.1)^2\right)^{\gamma/(\gamma-1)} \approx \frac{p_a}{.46835}$$  \hspace{1cm} (23a)

$$p_0 = p_a \left(1 + \frac{\gamma - 1}{2}(1.2)^2\right)^{\gamma/(\gamma-1)} \approx \frac{p_a}{.41238}$$  \hspace{1cm} (23b)

Again, the ratios are accurate given $\gamma = 1.4$.

An additional consideration, absent from the analysis of the sonic nozzle but critical to the supersonic nozzles is that of area ratios. The ratio of exit area to throat area $A_e/A_t$ is directly tied to exit Mach number. The ratios can be determined from the principle of a constant mass...
flow rate between the throat and the nozzle exit:

\[ \dot{m} = \rho AV \]

\[ \dot{m}_t = \dot{m}_e \]

\[ \rho_t A_t V_t = \rho_e A_e V_e \]

Using Eq. (5) and (7), and plugging in \( M_t = 1 \) this becomes:

\[
A_t p_0 \left( \frac{\gamma}{RT_0} \left( 1 + \frac{\gamma - 1}{2} \right) \right)^{-\frac{\gamma+1}{2\gamma - 1}} = A_e p_0 \left( \frac{\gamma}{RT_0} M_e \left( 1 + \frac{\gamma - 1}{2} M_e^2 \right) \right)^{-\frac{\gamma+1}{2\gamma - 1}}
\]

(24)

Cancellation of superfluous terms and rearrangement yields:

\[
\frac{A_e}{A_t} = \frac{1}{M_e} \left[ \left( \frac{2}{\gamma + 1} \right) \left( 1 + \frac{\gamma - 1}{2} M_e^2 \right) \right]^{\frac{\gamma+1}{2\gamma - 1}}
\]

(25)

Due to the geometry of the nozzle, \( A = 2h \tau \) is directly proportional to \( h \), therefore \( A_e/A_t = h_e/h_t \). By recalling that any half-height normalized by \( h_t \) is simply a \( y \) value, Eq. 26 can be rewritten as:

\[
y_e = \frac{1}{M_e} \left[ \left( \frac{2}{\gamma + 1} \right) \left( 1 + \frac{\gamma - 1}{2} M_e^2 \right) \right]^{\frac{\gamma+1}{2\gamma - 1}}
\]

(26)

Approximate values for each nozzle, assuming \( \gamma = 1.4 \) are:

\[
y_e \approx \begin{cases} 
1.00793, & M_e = 1.1 \\
1.03044, & M_e = 1.2 
\end{cases}
\]

(27)

6 Probe Loading

A significant problem in the calibration of aerodynamic probes is posed by the aerodynamic loading they are subject to. When high velocity flows encounter probes they exert an aerodynamic force that causes a deflection of the probe tip. This is problematic in that the calibration process relies on a very specific understanding of the orientation of both the probe and the flowfield. When aerodynamic probes are placed in supersonic flow, a detached bow shock wave appears in front of the probe tip. This makes calculation of aerodynamic loading simpler in the sense that the physical flow past the probe will have been decelerated to a subsonic velocity.

6.1 Loading

If the assumption is made that the deflection angle will be small, the loading can be approximated by a load \( F_a \) along the longitudinal axis of the probe itself and a distributed load \( F_b \) along the length of the spur that connects the probe proper to the traversing rig. This description can be observed in Figure 11 below. In order to place the probe in the exact center of the flow, avoiding the boundary layer as much as possible, the length \( L_2 \) should be equivalent to \( h_e \).

Assuming the probe has constant radius \( r \), the loads can be approximated as:

\[
F_a = \frac{1}{2} \rho V^2 C_{Da} A_a
\]

(28a)

\[
F_b = \frac{1}{2} \rho V^2 C_{Da} A_b = w(L_2 - r)
\]

(28b)

The dynamic pressure - using Eq. 6 and Eq. 8 - can be expressed as a function of Mach number and stagnation pressure by:

\[
\frac{1}{2} \rho V^2 = \frac{\gamma}{2} p_0 M^2 \left( 1 + \frac{\gamma - 1}{2} M^2 \right)^{-\gamma/(\gamma - 1)}
\]

(29)
The areas $A_a$ and $A_b$ are the cross sections of the probe cone and cylindrical stem, respectively:

$$A_a = \pi r^2 \tag{30a}$$
$$A_b = 2r(L_2 - r) \tag{30b}$$

Figure 11: Diagram providing description of loading parameters (not to scale)

The final terms in Eq. 28a and 28b are more difficult to evaluate. First, drag coefficient is generally a function of Reynold’s number. In lieu of actual operating numbers for the wind tunnel in question, a Fermi approximation can be used to find a value for $Re$ within an order of magnitude.

$$Re = \frac{\rho V D}{\mu}$$
$$\rho \approx 10^0 \text{ [kg/m}^3\text{]}$$
$$V \approx 10^3 \text{ [m/s]}$$
$$D = 2r \approx 10^{-3} \text{ [m]}$$
$$\mu \approx 10^{-5} \text{ [kg/m-s]}$$

According to this approximation the Reynold’s number is on the order of $Re = 10^5$. Figure 16 in Appendix A shows $C_D$ vs. $Re$ for flow past a smooth cylinder. For the approximated $Re$, the drag coefficient is approximately $C_{D_b} = 1.2$ This value has been shown to drop significantly for cylinders of lower aspect ratio, to values of $C_{D_b} = 0.82$ for $AR = 10^{12}$. Based on the probe description of Gomes\textsuperscript{11}, depicted in Figure 12 in Appendix A, the cone angle is assumed to be $60^\circ$, giving $C_{D_a} \approx 0.8$ as well.\textsuperscript{12}
6.2 Deflection

If the probe is treated as a cylindrical beam, cantilevered at the origin at the traverse unit the loading can be characterized using singularity functions by:

\[ EI \frac{d^4 y}{dx^4} = (F_a + F_b)x^{-1} - w(x - L_1)^0 + w(x - (L_2 - r))^0 - F_a(x - (L_1 + L_2))^{-1} \]  

(31)

Where \( w = F_b/(L_2 - r) \), E is Young’s modulus, and I is the area moment of inertia:

\[ I = \frac{m}{48} (12r^2 + 4(L_1 + L_2)^2) \]

with, \( m = \pi r^2 (L_1 + L_2) \rho_{\text{probe}} \)

Using the boundary conditions of a cantilevered beam: \( y = 0 \) and \( \frac{dy}{dx} = 0 \) at \( x = 0 \), Eq. 31 can be differentiated to determine the slope and the deflection at Pt. 1 on Figure 11:

At \( x = L_1 + L_2 \):

\[ \beta = \frac{dy_1}{dx} = \frac{1}{2EI} \left[ (F_a + F_b)(L_1 + L_2)^2 + \frac{w}{3} ((L_1 + r)^3 - L_2^3) \right] \]  

(32)

\[ y_1 = \frac{1}{6EI} \left[ (F_a + F_b)(L_1 + L_2)^3 + \frac{w}{4} ((L_1 + r)^4 - L_2^4) \right] \]  

(33)

Given the results of Eq. 32 and 33, the deflection at Pt. 2 can be determined through a few approximations. First, if we assume that the part of the probe parallel to the flow experiences very little aerodynamic loading, then the angle it makes with the probe stem can be assumed to remain at 90°. This symmetry places the rightward slope at 1 equal to the downward slope at 2: the probe will face an angle \( \beta \) below the center of the flow. Using this geometry - which is depicted in Figure 17 in Appendix A - the following can be determined:

\[ \frac{dy_2}{dx} = \beta = \frac{dy_1}{dx} \]

(34)

\[ \tan \beta = \frac{y_1}{L_1 + L_2} \]

\[ \sin \beta = \frac{y_2}{L_3} \]

So: \( y_2 = y_1 \cos \beta \left( \frac{L_3}{L_1 + L_2} \right) \)

(35)

Eq. 34 and 35 provide tip deflection and tip angle of the probe under the supposed loading described previously. This analysis certainly leaves a lot of work to provide a numerical solution. This Section merely serves as a guide for the thought process involved in determining the deflection of the probe tip in this application.

7 Conclusion

This report provides analytical solutions to the problem of designing nozzle contours to produce flows at Mach numbers of 1.0, 1.1, and 1.2. It also provides theory behind these solutions as well as suggestions on how to incorporate them into an existing wind tunnel rig. Additionally, a brief discussion of the loading considerations for probes placed in the flows is provided. Full documents, both of the sources cited in this report as well as other, relevant reports can be found via the link provided in Appendix B.
8 References

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11. Evvard, J.C., and Marcus, L.R.; 1952
    Achievement of Continuous Wall Curvature in Design of Two-Dimensional Symmetrical Supersonic Nozzles - NACA Technical Note 2616.

12. Shaughnessy, E.J.; 2005
    "Introduction to Fluid Mechanics" - Oxford University Press.

A Additional Figures

Figure 12: Example diagram of a 5-hole aerodynamic probe\textsuperscript{[1]}

Figure 13: Diagram of existing calibration rig nozzle inserts\textsuperscript{[1]}

Figure 14: Diagram of nozzle characteristics with continuous curvature\textsuperscript{[1]}

\bf{\textsuperscript{[1]}} Reference text.
Figure 15: Example of POI discontinuity using method of Foelsch\textsuperscript{[7]}

Figure 16: Plot of $C_D$ vs. $Re$ for flow past a smooth cylinder\textsuperscript{[12]}
Additional Reading

The documents cited herein as well as additional relevant documents can be found in a publicly accessible folder on Google Drive at either of the following URLs:

https://drive.google.com/folderview?id=0B8EjCZerEw8lczUtU0ZLaEtxZjA&usp=sharing

http://bit.ly/1kiXUJh

The documents are in .pdf format and will be accessible indefinitely to any person with this link.