Energy Flux Vector Derivation

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To find the energy flux vector of a wave, we start with the force vector, introduce perturbations, and average over a cycle. This sounds simple but the process quickly becomes very tedious. Starting with the force vector:

$$\mathbf{F} = \rho E\mathbf{U} + p\mathbf{U} \tag{1}$$

Next we have to introduce perturbations. Our four base quantities are pressure (p), density (ρ), and the two velocity vectors u and v. Only perturbations in these quantities may be considered, because steady-state values of more complex quantities are not necessarily equal to the product of the steady state base quantities, due to the nature of integrating products of sinusoidal equations. Assuming sinusoidal perturbations:

$$\rho = \bar{\rho} + \rho_r cos(\omega t) - \rho_i sin(\omega t) \tag{2}$$

$$p = \bar{p} + p_r cos(\omega t) - p_i sin(\omega t) \tag{3}$$

$$u = \bar{u} + u_r cos(\omega t) - u_i sin(\omega t) \tag{4}$$

$$v = \bar{v} + v_r cos(\omega t) - v_i sin(\omega t) \tag{5}$$

$$\mathbf{U} = u\mathbf{i} + v\mathbf{j} \tag{6}$$

Energy becomes a little trickier to deal with. The isentropic energy density is considered:

$$\rho e = \frac{p}{(\gamma - 1)} \tag{7}$$

Assuming isentropic flow, pressure changes adiabatically as density:

$$p = k\rho^\gamma \tag{8}$$

Taking the derivative of pressure with respect to density, we get:

$$\frac{\partial p}{\partial \rho} = k\gamma\rho^{\gamma-1} = \frac{k\gamma\rho^\gamma}{\rho} = \frac{\gamma p}{\rho} \tag{9}$$

Similarly, taking the second derivative, we get:

$$\frac{\partial^2 p}{\partial \rho^2} = k\gamma(\gamma-1)\rho^{\gamma-2} = \frac{k\gamma(\gamma-1) * \rho^\gamma}{\rho^2} = \frac{\gamma(\gamma-1)p}{\rho^2} \tag{10}$$

The above calculations are important for the following: perturbations in pressure are dependent on perturbations in density, allowing a second-order Taylor expansion of the pressure in terms of the density about the average values of pressure and density:

$$p = p_0 + \frac{\partial p}{\partial \rho}|_0 \Delta\rho + \frac{1}{2}\frac{\partial^2 p}{\partial \rho^2}|_0 \Delta\rho^2 \tag{11}$$

Substituting in equations (9) and (10) and evaluating about the average, the result is:

$$p = \bar{p} + \frac{\gamma\bar{p}}{\bar{\rho}}\Delta\rho + \frac{\gamma(\gamma-1)\bar{p}}{2\bar{\rho}^2}\Delta\rho^2 \tag{12}$$

Inserting this into equation (7) to solve for the internal energy, one gets:

$$\rho e = \frac{\bar{p}}{\gamma-1} + \frac{\gamma\bar{p}}{\bar{\rho}(\gamma-1)}\Delta\rho + \frac{\gamma\bar{p}}{2\bar{\rho}^2}\Delta\rho^2 \tag{13}$$

Finally, there is a beautiful simplification that can be made. The speed of sound can be expressed as:

$$c = \sqrt{\frac{\partial p}{\partial \rho}} \tag{14}$$

Inserting equation (9) and squaring, one gets:

$$c^2 = \frac{\gamma\bar{p}}{\bar{\rho}} \tag{15}$$

This allows us to simplify equation (13) for internal energy:

$$\rho e = \frac{\bar{p}}{\gamma-1} + \frac{c^2}{(\gamma-1)}\Delta\rho + \frac{c^2}{2\bar{\rho}}\Delta\rho^2 \tag{16}$$

Total energy per unit volume is the sum of internal and kinetic energy:

$$\rho E = \rho e + \frac{1}{2}\rho(u^2+v^2) \tag{17}$$

The kinetic energy is found by adding perturbations to the velocity components and the density component, then expanding. The third-order term is neglected:

$$\frac{1}{2}\rho(u^2+v^2) = \frac{1}{2}\bar{\rho}(\bar{u}^2+\bar{v}^2)+\frac{1}{2}\Delta\rho(\bar{u}^2+\bar{v}^2)+\bar{\rho}(\bar{u}\Delta u+\bar{v}\Delta v)+\Delta\rho(\bar{u}\Delta u+\bar{v}\Delta v)+\frac{1}{2}\bar{\rho}(\Delta u^2+\Delta v^2) \tag{18}$$

The total energy, including perturbations, is found by adding together the internal and kinetic energy. First order terms are immediately neglected, because they average out to 0 and as such cannot contribute to the energy flux.

So the final total energy term is found by combining equations (16), (17) and (18) and dropping terms with first order perturbations. Assuming sinusoidal perturbations, the final term becomes:

$$\rho E = \frac{\bar{p}}{\gamma-1} + \frac{1}{2}\bar{\rho}(\bar{u}^2+\bar{v}^2) + \frac{c^2}{(\gamma-1)}\Delta\rho + \frac{c^2}{2\bar{\rho}}\Delta\rho^2 + \frac{1}{2}\bar{\rho}(\Delta u^2+\Delta v^2) \tag{19}$$

$$\rho E = \frac{\bar{p}}{\gamma - 1} + \frac{1}{2}\bar{\rho}(\bar{u}^2 + \bar{v}^2) + \frac{c^2}{(\gamma - 1)}(\rho_r cos(\omega t) - \rho_i sin(\omega t)) + \frac{c^2}{2\bar{\rho}}(\rho_r cos(\omega t) - \rho_i sin(\omega t))^2$$
$$+ \frac{1}{2}\bar{\rho}((u_r cos(\omega t) - u_i sin(\omega t))^2 + (v_r cos(\omega t) - v_i sin(\omega t))^2) \quad (20)$$

Substituting into the force equation, one gets many terms in total... However, after integrating over a cycle, most of the terms drop out. The details are omitted for brevity, but the process consists of integrating the force equation, with all its terms, over one cycle, and averaging by the period to get the time-average energy transfer vector. Lastly, the steady state terms are subtracted off, because we are interested only in the unsteady energy flux. The final result has 8 terms, easily handleable:

$$\bar{\mathbf{F}} = \frac{\omega}{2 * \pi} \int_0^{\frac{2\pi}{\omega}} \mathbf{F} dt$$
$$= \frac{1}{2}[p_i \mathbf{U}_i + p_r \mathbf{U}_r + \frac{1}{2}\rho \bar{\mathbf{U}}(u_r^2 + u_i^2) + \frac{1}{2}\rho \bar{\mathbf{U}}(v_r^2 + v_i^2) + \frac{c^2 \bar{\mathbf{U}}(\rho_r^2 + \rho_i^2)}{2\bar{\rho}}] \quad (21)$$

This result is difficult to understand. To compare to 2D theory, we will consider a 1-D cut-on acoustic wave. Firstly, everything is in phase, so all the imaginary terms go away. Such a wave also has the following properties:

$$\begin{pmatrix} \rho_r \\ u_r \\ v_r \\ p_r \end{pmatrix} = \begin{pmatrix} \rho\, c(\omega + v\, l) - \rho\, u\, S \\ -c\, u(\omega + v\, l) + c^2\, S \\ -c\, l(c^2 - u^2) \\ \rho\, c^3\, (\omega + v\, l) - \rho\, c^2\, u\, S \end{pmatrix} \quad (22)$$

Due to this being an acoustic wave, l goes to 0, and S goes to ω. Equation(13) becomes:

$$\begin{pmatrix} \rho_r \\ u_r \\ v_r \\ p_r \end{pmatrix} = \begin{pmatrix} \bar{\rho} \\ c \\ 0 \\ \bar{\rho}\, c^2 \end{pmatrix} \quad (23)$$

To determine which terms are negligible in the final equation, we introduce a multiplier α to the vector in equation (12); any term with α to an order of 3 or greater is neglected. It turns out that every term has an α^2 in it, so no terms are neglected. There is therefore no need to retain the α.

Finally, by definition:

$$\mathbf{U}_r = u_r \mathbf{i} + v_r \mathbf{j} = c\mathbf{i} \quad (24)$$

With all these assumptions in mind, equation (11) becomes:

$$\overline{\mathbf{F}} = \frac{\bar{\rho}c^2}{2}[\overline{\mathbf{U}} + c\mathbf{i}] \tag{25}$$

This is exactly what one would expect. To understand intuitively what the answer should be, one has to understand the concept of *group velocity*, or the velocity at which the wave envelope travels through space. In the case of a 1-D acoustic wave, it can be shown that the group velocity is equal to $\overline{\mathbf{U}} + c\mathbf{i}$. This makes intuitive sense; the velocity of a 1-D acoustic wave in a flow is equal to the velocity at which the flow is moving plus the speed of sound in the direction of wave propagation. Group velocity is usually associated with the direction of energy transport, at least in a non-dispersive medium, so the energy transport vector should align with the group velocity vector, as it does (with a multiplier).

To look at the validity of this theory, specifically how closely it aligns with the group velocity, we will consider the case of a 1D cut-on wave produced by a 1D flow that is perpendicular to the wave. In this case, the flow speed is Mc, in the j-direction. We will look at the direction of energy propagation by plotting direction vs. Mach number, the direction being found by dividing the j-vector by the i-vector. According to group velocity, the direction should be 45° when the Mach number is 1, 0° when the Mach number is 0, and linear. We will consider: $\bar{u}$ becomes 0 and $\bar{v}$ becomes Mc, so $\overline{\mathbf{U}}$ becomes $Mc\mathbf{j}$. Equation (14) reduces to:

$$\overline{\mathbf{F}} = \frac{\bar{\rho}c^3}{2}[M\mathbf{j} + \mathbf{i}] \tag{26}$$

The direction is then found by:

$$Direction = arctan(M) \tag{27}$$

And it works! When M equals 1, the direction is equal to 45 degrees.

Note 1: The changes made to this theory from Paul's work are: the adiabatic pressure-density coupled expansion, taking ρE as a unit instead of just E, and dropping the second-order terms from the energy vector instead of the final force vector.

Note 2: This should still reconcile with Paul's theory, but for the life of me I cannot get it to work his way. Good luck to whoever works on this next!